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Quantum Larmor radiation from a moving charge in an electromagnetic plane wave background

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We extend our previous work [Phys. Rev. D**83** 045030 (2011)], which investigated the first-order quantum effect in the Larmor radiation from a moving charge in a spatially homogeneous time-dependent electric field. Specifically, we investigate the quantum Larmor radiation from a moving charge in a monochromatic electromagnetic plane wave background based on the scalar quantum electrodynamics at the lowest order of the perturbation theory. Using the in-in formalism, we derive the theoretical formula of the total radiation energy from a charged particle in the initial states being at rest and being in a relativistic motion. Expanding the theoretical formula in terms of the Planck constant $\hbar$, we obtain the first-order quantum effect on the Larmor radiation. The quantum effect generally suppresses the total radiation energy compared with the prediction of the classical Larmor formula, which is a contrast to the previous work. The reason is explained by the fact that the radiation from a moving charge in a monochromatic electromagnetic plane wave is expressed in terms of the inelastic collisions between an electron and photons of the background electromagnetic waves.

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1. Introduction

A series of the previous works[1,2,3,4] focused the investigation on the quantum effect on the Larmor radiation from a charge in an accelerated motion on a non-trivial background field. These investigations were partially motivated by a series of Higuchi and Martin's work[5,6,7], which derived the radiation reaction force on a charged particle in an accelerated motion on the basis of the quantum electrodynamics (QED). The Larmor radiation from a moving charge in an accelerated motion is well-known in the classical electrodynamics[8], and Higuchi and Martin showed how it can be reproduced in the limit of $\hbar \to 0$, where $\hbar$ is the Planck constant, in the framework of the scalar quantum electrodynamics (SQED).

Higuchi and Martin's works led the study on the quantum correction to the classical Larmor radiation using their theoretical framework. In Ref.1, Higuchi and Walker investigated the quantum correction to the classical Larmor radiation from a moving charge whose motion is non-relativistic. They found that the quantum correction is non-local and that the quantum effect suppresses the total radiation when the particle is in a non-relativistic motion. In our previous work[2], we extended the work by Higuchi and Walker to the case of a moving charge in a relativistic motion, assuming a spatially homogeneous time-dependent electric field background. We found that the quantum effect may increase the total radiation energy compared with the classical evaluation when the particle is in a relativistic motion[2].

In the previous works, however, it has not been clearly understood what determines the decrease or increase of the total radiation energy of the quantum Larmor radiation. The results in the previous works might rely on the assumption of the spatially homogeneous electric field background. Further investigation in more general background might give us a hint to understand the origin of the quantum correction to the Larmor radiation. Then, as an extension of the previous works, we consider the case of a monochromatic electromagnetic plane wave background in the present paper. We evaluate the radiation energy in the framework of the SQED using the solution for the complex scalar field equation on a monochromatic electromagnetic plane wave, which is a counterpart of the Volkov solution for the Dirac equation[9].

There have been many works on the radiation process from a moving charge in an electromagnetic plane wave background so far (e.g., Ref.10, 11, 12). However, our work in the present paper is different from the previous works in the following points. First, we derive the theoretical formula for the radiation energy starting with the theoretical framework of the in-in formalism[13]. Second, we demonstrate that the quantum effect generally suppresses the total radiation energy by explicitly evaluating the contribution at the order of $\hbar$. We also demonstrate that the quantum effect of the order of $\hbar$ is explained *only* by that the radiation process is expressed in terms of the inelastic collisions between an electron and photons of the monochromatic electromagnetic wave background.

The reason why we should use the in-in formalism is explained as follows. The in-in formalism is useful to avoid the subtle problem of the definition of the vacuum state for charged particles. For example, when we consider a charged particle in a constant electric field, we should choose the different vacuum state for the initial state at the infinitely past time and the final state at infinitely future time[14]. Then, the radiation process due to interaction from the charged particle must be carefully treated[15]. The in-in formalism formulated by Weinberg[13], which was developed to evaluate higher order correlation functions in an inflationary universe, is useful to treat this problem[3,2].

It is known that a charged particle moving in a curved space-time generally gives rise to radiation, and the classical radiation reaction force was first derived by De Witt and Brehme[16], and Hobbs[17]. The radiation reaction force in the conformally flat universe is also investigated in Ref.18. Then, several authors investigated

the quantum effect on the radiation from a moving charged particle in an expanding universe[19,4,20]. It is demonstrated that the in-in formalism is useful to investigate the quantum effect on the radiation from a moving charge in an expanding universe[3], because the vacuum state for charged particles is changed there. The authors in Ref.3 demonstrated that the quantum effect suppresses the total radiation energy in comparison with the classical counterpart (see also Ref.4).

This paper is organized as follows. In section 2, we derive a general formula for the quantum radiation from a charged particle in a monochromatic electromagnetic plane wave background based on the SQED. We consider the initial states that the incident charged particle is at rest and in a relativistic motion along with the direction of the polarization vector. In section 3, we present the explicit formulas for the radiation with these initial states. Then, we discuss about the origin of the quantum effect by extracting the terms of the order of $\hbar$. Section 4 is devoted to summary and conclusions. Throughout this paper, we adopt the unit in which the speed of light equals unity $c = 1$, and the metric convention $(-, +, +, +)$.

2. formulation

In this section, we derive the formula for the radiation energy from a moving charge in an electromagnetic plane wave background. We consider the scalar QED with the action,

$$S = \int d^4x \left[-\eta^{\mu\nu} \left(\partial_\mu - \frac{ie}{\hbar} A_\mu \right) \phi^\dagger \left(\partial_\nu + \frac{ie}{\hbar} A_\nu \right) \phi - \frac{m^2}{\hbar^2} \phi^\dagger \phi - \frac{1}{4\mu_0} F^{\mu\nu} F_{\mu\nu} \right], \quad (1)$$

where e and m are the charge and mass of the complex scalar field ϕ, respectively, $\eta^{\mu\nu}$ is the metric of the Minkowski space-time, A^μ is the electromagnetic field, the field strength is defined by $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$, and μ_0 is the magnetic permeability of vacuum. We work in the Minkowski space-time, but consider the electromagnetic plane wave background.

The equation of motion of the complex scalar field is written

$$\left[\left(\partial_\mu + \frac{ie}{\hbar} \bar{A}_\mu \right) \left(\partial^\mu + \frac{ie}{\hbar} \bar{A}^\mu \right) - \frac{m^2}{\hbar^2} \right] \phi = 0, \quad (2)$$

where $\bar{A}^\mu$ is the background electromagnetic field. The quantized field is written as

$$\phi = \sum_{\mathbf{p}} \sqrt{\frac{\hbar}{V}} (\phi_{\mathbf{p}} b_{\mathbf{p}} + \phi^*_{\mathbf{p}} c^\dagger_{\mathbf{p}}), \quad (3)$$

where V is the volume of a box, $b_{\mathbf{p}}$ and $c^\dagger_{\mathbf{p}}$ are the annihilation operator of the particle and the creation operator of the anti-particle, respectively, and $\phi_{\mathbf{p}}$ is the mode function decomposed in the finite box. These operators satisfy the commutation relation, $[b_{\mathbf{p}}, b^\dagger_{\mathbf{p}'}] = \delta_{\mathbf{p},\mathbf{p}'}$, $[c_{\mathbf{p}}, c^\dagger_{\mathbf{p}'}] = \delta_{\mathbf{p},\mathbf{p}'}$, and the other combinations are zero, and the vacuum state is defined by $b_{\mathbf{p}} |0\rangle_{b_{\mathbf{p}}} = 0$, and $c_{\mathbf{p}} |0\rangle_{c_{\mathbf{p}}} = 0$. We assume that the background electromagnetic field depends only on $k \cdot x$, where the dot denotes the four dimensional contraction, $k \cdot x \equiv k^\mu x_\mu$. We adopt the Lorentz gauge condition,

i.e., $\partial^\mu \bar{A}_\mu = 0$, where $\bar{A}_\mu = \bar{A}_\mu(k \cdot x)$. In this situation, we can solve the equation of motion as follows (cf. Ref.9). First, we rewrite the equation of motion as

$$\left(\partial_\mu \partial^\mu + 2\frac{ie}{\hbar}\bar{A}_\mu \partial^\mu - \frac{e^2}{\hbar^2}\bar{A}^2 - \frac{m^2}{\hbar^2}\right)\phi_{\mathbf{p}} = 0, \tag{4}$$

where $\bar{A}^2 \equiv \bar{A} \cdot \bar{A}$. We seek the solution of the form,

$$\phi_{\mathbf{p}} = \sqrt{\frac{\hbar}{2p^0}} e^{ip\cdot x/\hbar} g_p(k \cdot x), \tag{5}$$

with the initial condition, $\phi_{\mathbf{p}} \sim \sqrt{\frac{\hbar}{2p^0}} e^{ip\cdot x/\hbar}$, at $t \to -\infty$. Using the relations $k^2 = 0$, $\partial^\mu \bar{A}_\mu = 0$ and $p^2 = -m^2$, equation for $g_p(k \cdot x)$ becomes

$$\frac{dg_p(k\cdot x)}{d(k\cdot x)} + \frac{i}{2\hbar p \cdot k}\left(2ep\cdot\bar{A} + e^2\bar{A}^2\right) g_p(k\cdot x) = 0. \tag{6}$$

Then, we find the solution

$$\phi_{\mathbf{p}} = \sqrt{\frac{\hbar}{2p^0}} e^{ip\cdot x/\hbar} \exp\left[-i\int_{-\infty}^{k\cdot x} \frac{1}{2\hbar p\cdot k}[2ep\cdot\bar{A}((k\cdot x)') + e^2\bar{A}^2((k\cdot x)')]d(k\cdot x)'\right]. \tag{7}$$

On the other hand, the quantized fluctuations of the electromagnetic field can be described

$$A_\mu = \sqrt{\frac{\mu_0 \hbar}{V}} \sum_{(\lambda),\mathbf{k}'} \epsilon_\mu^{(\lambda)} \sqrt{\frac{1}{2k'}} e^{i\mathbf{k}'\cdot\mathbf{x}} (a_{\mathbf{k}'}^{(\lambda)} e^{-ikt} + a_{-\mathbf{k}'}^{\dagger(\lambda)} e^{ikt}), \tag{8}$$

where $\epsilon_\mu^{(\lambda)}$ is the polarization vector, $a_{\mathbf{k}'}^{\dagger(\lambda)}$ and $a_{\mathbf{k}'}^{(\lambda)}$ are the creation and annihilation operator of the electromagnetic field, respectively, which satisfy the commutation relation $[a_{\mathbf{k}}^{(\lambda)}, a_{\mathbf{k}'}^{\dagger(\lambda')}] = \delta_{(\lambda),(\lambda')}\delta_{\mathbf{k},\mathbf{k}'}$, $[a_{\mathbf{k}}^{(\lambda)}, a_{\mathbf{k}'}^{(\lambda')}] = [a_{\mathbf{k}}^{\dagger(\lambda)}, a_{\mathbf{k}'}^{\dagger(\lambda')}] = 0$ and the vacuum state is defined by $a_{\mathbf{k}}^{(\lambda)} |0\rangle_{a_{\mathbf{k}}} = 0$.

We consider the lowest-order contribution of the process so that one photon is emitted from a charged particle with the initial momentum $\mathbf{p}_i$, as shown in Fig. 1, whose interaction Hamiltonian is

$$H_I = -\frac{ie}{\hbar}\int d^3\mathbf{x} A^\mu \left\{(\partial_\mu \phi^\dagger)\phi - \phi^\dagger(\partial_\mu \phi) - \frac{2ie}{\hbar}\bar{A}_\mu \phi^\dagger \phi\right\}. \tag{9}$$

Substituting Eq. (7) into Eq. (9), we obtain

$$\begin{aligned} :H_I: = -e\int d^3\mathbf{x}\sqrt{\frac{\mu_0}{4\hbar V^3}} \sum_{(\lambda),\mathbf{k}',\mathbf{p}_1,\mathbf{p}_2} \epsilon^{(\lambda)\mu} \frac{1}{\sqrt{2k' p_1^0 p_2^0}} e^{i\mathbf{k}'\cdot\mathbf{x}} (a_{\mathbf{k}'}^{(\lambda)} e^{-ikt} + a_{-\mathbf{k}'}^{\dagger(\lambda)} e^{ikt}) \\ \times \left(I_{p_1\mu} + I_{p_2\mu} + 2e\bar{A}_\mu\right) g_{p_1}^* g_{p_2} e^{-i(p_1-p_2)\cdot x/\hbar} b_{\mathbf{p}_1}^\dagger b_{\mathbf{p}_2}, \end{aligned} \tag{10}$$

where we defined

$$I_{p\mu} \equiv p_\mu - \frac{k_\mu}{2p\cdot k}(2ep\cdot\bar{A} + e^2\bar{A}^2). \tag{11}$$

In order to derive the total radiation energy, we first evaluate the expectation value of the number operator[3,2]. Using the in-in formalism[13,21], we may write the expectation value of the number operator of the emitted photon with the wavenumber $\mathbf{k}_f$, up to the second-order of the perturbative expansion, as follows,

$$
\begin{aligned}
\sum_{(\lambda)} \left\langle a^{\dagger(\lambda)}_{\mathbf{k}_f} a^{(\lambda)}_{\mathbf{k}_f} \right\rangle &= \sum_{(\lambda)} \frac{1}{\hbar^2} \mathrm{Re} \left[\int_{-\infty}^{\infty} dt_2 \int_{-\infty}^{\infty} dt_1 \left\langle H_I(t_1) a^{\dagger(\lambda)}_{\mathbf{k}_f} a^{(\lambda)}_{\mathbf{k}_f} H_I(t_2) \right\rangle \right], \\
&= \mathrm{Re}\Big[\frac{\mu_0 e^2}{8\hbar V^3} \int_{-\infty}^{\infty} dt_2 \int_{-\infty}^{\infty} dt_1 \int d^3\mathbf{x}_1 \int d^3\mathbf{x}_2 \\
&\qquad \sum_{(\lambda),(\lambda_1),\mathbf{k}_1,(\lambda_2),\mathbf{k}_2} \ \sum_{\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3,\mathbf{p}_4} \epsilon^{(\lambda_1)}_{\mu} \epsilon^{(\lambda_2)}_{\nu} \frac{1}{\sqrt{k_1 k_2 p_1^0 p_2^0 p_3^0 p_4^0}} \\
&\times \langle \mathrm{in} | \, (a^{(\lambda_1)}_{\mathbf{k}_1} e^{-ikt_1} + a^{\dagger(\lambda_1)}_{-\mathbf{k}_1} e^{ikt_1}) a^{\dagger(\lambda)}_{\mathbf{k}_f} a^{(\lambda)}_{\mathbf{k}_f} \\
&\qquad (a^{(\lambda_2)}_{\mathbf{k}_2} e^{-ikt_2} + a^{\dagger(\lambda_2)}_{-\mathbf{k}_2} e^{ikt_2}) b^{\dagger}_{\mathbf{p}_1} b_{\mathbf{p}_2} b^{\dagger}_{\mathbf{p}_3} b_{\mathbf{p}_4} \, | \mathrm{in} \rangle \\
&\times e^{-i(\mathbf{p}_1 - \mathbf{p}_2 - \hbar\mathbf{k}_1)\cdot\mathbf{x}_1/\hbar} e^{-i(\mathbf{p}_3 - \mathbf{p}_4 - \hbar\mathbf{k}_2)\cdot\mathbf{x}_2/\hbar} e^{i(p_{10} - p_{20})t_1/\hbar} e^{i(p_{30} - p_{40})t_2/\hbar} \\
&\times (I^{\mu}_{p_1} + I^{\mu}_{p_2} + 2e\bar{A}^{\mu})_{x_1} (I^{\nu}_{p_3} + I^{\nu}_{p_4} + 2e\bar{A}^{\nu})_{x_2} (g^*_{p_1} g_{p_2})_{x_1} (g^*_{p_3} g_{p_4})_{x_2} \Big].
\end{aligned}
\tag{12}
$$

We adopt the initial state as the one-particle state with the momentum $\mathbf{p}_i$, which

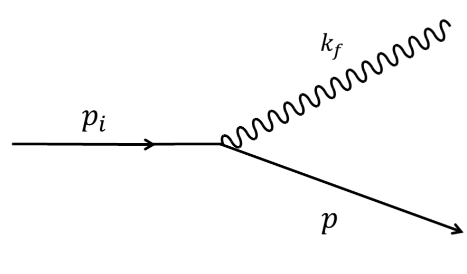

Fig. 1. Feynman diagram for the process.

is described by $|\mathrm{in}\rangle = |0\rangle_{a^{(\lambda)}_{\mathbf{k}}} \otimes b^\dagger_{\mathbf{p}_\mathrm{i}}|0\rangle_{b_\mathbf{p}} \otimes |0\rangle_{c_\mathbf{p}}$, then we have

$$\sum_{(\lambda)} \left\langle a^{\dagger(\lambda)}_{\mathbf{k}_f} a^{(\lambda)}_{\mathbf{k}_f} \right\rangle = \mathrm{Re}\Big[\frac{\mu_0 e^2}{8\hbar V^3}\int_{-\infty}^{\infty} dt_2 \int_{-\infty}^{\infty} dt_1 \int d^3\mathbf{x}_1 \int d^3\mathbf{x}_2$$
$$\sum_{\mathbf{p}} \frac{1}{k_f p_\mathrm{i}^0 p^0} e^{-i(p_\mathrm{i}-p-\hbar k_f)\cdot x_1/\hbar} e^{-i(p-p_\mathrm{i}+\hbar k_f)\cdot x_2/\hbar}$$
$$\times (I^\mu_{p_\mathrm{i}} + I^\mu_p + 2e\bar{A}^\mu)_{x_1} (I_{p_\mathrm{i}\mu} + I_{p\mu} + 2e\bar{A}_\mu)_{x_2} (g^*_{p_\mathrm{i}} g_p)_{x_1} (g^*_p g_{p_\mathrm{i}})_{x_2}\Big]. \tag{13}$$

Note that $g^*_{p_\mathrm{i}} g_p$ gives the phase factor,

$$g^*_{p_\mathrm{i}} g_p = \exp\left[\frac{i}{\hbar}\int d(k\cdot x)\left\{e\left(\frac{p_\mathrm{i}\cdot\bar{A}}{p_\mathrm{i}\cdot k} - \frac{p\cdot\bar{A}}{p\cdot k}\right) + \frac{e^2\bar{A}^2}{2}\left(\frac{1}{p_\mathrm{i}\cdot k} - \frac{1}{p\cdot k}\right)\right\}\right]. \tag{14}$$

The vector potential of the monochromatic electromagnetic plane wave is written

$$\bar{A}_\mu = \frac{(a_\mu e^{ik\cdot x} + a^*_\mu e^{-ik\cdot x})}{2}, \tag{15}$$

for general arbitrary polarization, where a_μ is a complex constant vector to specify the amplitude and the polarization. We introduce dimensionless parameters as

$$\sigma_{\mathrm{i}p} = \frac{e^2}{4} a\cdot a^* \left(\frac{1}{p_\mathrm{i}\cdot k} - \frac{1}{p\cdot k}\right), \tag{16}$$

$$\xi_{\mathrm{i}p} e^{i\alpha_{\mathrm{i}p}} = e\left(\frac{p_\mathrm{i}\cdot a}{p_\mathrm{i}\cdot k} - \frac{p\cdot a}{p\cdot k}\right), \tag{17}$$

$$\eta_{\mathrm{i}p} e^{i\beta_{\mathrm{i}p}} = \frac{e^2}{4} a\cdot a \left(\frac{1}{p_\mathrm{i}\cdot k} - \frac{1}{p\cdot k}\right). \tag{18}$$

Note that $\eta_{\mathrm{i}p} = 0$ for the circular polarization while $\eta_{\mathrm{i}p} = \sigma_{\mathrm{i}p}, \alpha_{\mathrm{i}p} = \beta_{\mathrm{i}p} = 0$ for the linear polarization. We also introduce $\sigma_{p\mathrm{i}}(= -\sigma_{\mathrm{i}p})$, $\xi_{p\mathrm{i}}(= -\xi_{\mathrm{i}p})$, $\eta_{p\mathrm{i}}(= -\eta_{\mathrm{i}p})$, $\alpha_{p\mathrm{i}}(= \alpha_{\mathrm{i}p})$, and $\beta_{p\mathrm{i}}(= \beta_{\mathrm{i}p})$. Then, Eq. (14) reduces to

$$g^*_{p_\mathrm{i}} g_p = \exp\left[\frac{i}{\hbar}\int d(k\cdot x)\left(\sigma_{\mathrm{i}p} + \xi_{\mathrm{i}p}\cos(k\cdot x + \alpha_{\mathrm{i}p}) + \eta_{\mathrm{i}p}\cos(2k\cdot x + \beta_{\mathrm{i}p})\right)\right]$$
$$= \exp\left[\frac{i}{\hbar}\left(\sigma_{\mathrm{i}p} k\cdot x + \xi_{\mathrm{i}p}\sin(k\cdot x + \alpha_{\mathrm{i}p}) + \frac{\eta_{\mathrm{i}p}}{2}\sin(2k\cdot x + \beta_{\mathrm{i}p})\right)\right], \tag{19}$$

where we removed the divergent constant, which appears after the integration in deriving the second line. The divergent constant is a constant phase factor, which may related with the initial phase, and has no physical meaning. With the use of the mathematical formula

$$e^{iz\sin\theta} = \sum_{s=-\infty}^{\infty} e^{is\theta} J_s(z), \tag{20}$$

we rewrite Eq. (19) as

$$g_{p_{\rm i}}^* g_p = e^{i\sigma_{\rm ip} k\cdot x/\hbar} \sum_{\ell=-\infty}^{\infty} e^{i\ell k\cdot x} C_\ell\left(\xi_{\rm ip}/\hbar, \eta_{\rm ip}/\hbar, \alpha_{\rm ip}, \beta_{\rm ip}\right), \tag{21}$$

where we defined

$$C_\ell\left(\xi_{\rm ip}/\hbar, \eta_{\rm ip}/\hbar, \alpha_{\rm ip}\beta_{\rm ip}\right) = \sum_{s=-\infty}^{\infty} e^{i(\ell-2s)\alpha_{\rm ip}} e^{is\beta_{\rm ip}} J_{\ell-2s}\left(\xi_{\rm ip}/\hbar\right) J_s\left(\eta_{\rm ip}/2\hbar\right). \tag{22}$$

We can write

$$(I^\mu_{p_{\rm i}} + I^\mu_p + 2e\bar{A}^\mu)_{x_1}(I_{p_{\rm i}\mu} + I_{p\mu} + 2e\bar{A}_\mu)_{x_2} = \sum_{j,j'=-2}^{2} M_{j,j'} e^{ijk\cdot x_1} e^{ij'k\cdot x_2}, \tag{23}$$

where we define the kernel $M_{j,j'}$ as follows,

$$\begin{aligned}
M_{0,0} &= -2m^2 + 2p_{\rm i}\cdot p - \frac{e^2 a\cdot a^*}{2}(p_{\rm i}+p)\cdot k\left(\frac{1}{p_{\rm i}\cdot k} + \frac{1}{p\cdot k}\right),\\
M_{0,1} &= M_{1,0} = M^*_{0,-1} = M^*_{-1,0} = \frac{\xi_{\rm ip}}{2} e^{i\alpha_{\rm ip}}(p_{\rm i} - p)\cdot k,\\
M_{1,1} &= M^*_{-1,-1} = e^2 a\cdot a, \qquad M_{-1,1} = M^*_{1,-1} = e^2 a\cdot a^*\\
M_{0,2} &= M_{2,0} = M^*_{0,-2} = M^*_{-2,0} = -\frac{e^2}{8}(p_{\rm i}+p)\cdot k\left(\frac{1}{p_{\rm i}\cdot k} + \frac{1}{p\cdot k}\right) a\cdot a,
\end{aligned}$$

and the other components of $M_{j,j'}$ are zero. By performing the integration with respect to $\mathbf{x}_2$ in Eq. (13), we have the momentum conservation

$$\mathbf{p} = \mathbf{p}_{\rm i} - \hbar\mathbf{k}_f - \mathbf{k}[\sigma_{\rm ip} + (\ell + j)\hbar], \tag{24}$$

and, after summing over $\mathbf{p}$, Eq. (13) reduces to

$$\begin{aligned}
\sum_\lambda \left\langle a^{\dagger\lambda}_{k_f} a^\lambda_{k_f}\right\rangle = {\rm Re}\Big[&\frac{\mu_0 e^2}{8V\hbar}\int_{-\infty}^{\infty} dt_2 \int_{-\infty}^{\infty} dt_1\\
&\sum_{\ell,\ell',j,j'} \frac{1}{k_f p_i^0 p^0} M_{j,j'} C_\ell(\xi_{\rm ip}/\hbar, \eta_{\rm ip}/\hbar, \beta_{\rm ip}) C_{\ell'}(\xi_{\rm pi}/\hbar, \eta_{\rm pi}/\hbar, \beta_{\rm ip})\\
&\times \frac{1}{V}\int d^3\mathbf{x}_1 e^{-i\hbar\mathbf{k}(\ell+j+\ell'+j')\cdot\mathbf{x}_1/\hbar}\\
&\times e^{i(p_{0{\rm i}} - p_0 - \hbar k_f - k[\sigma_{\rm ip} + (\ell+j)\hbar])\cdot t_1/\hbar} e^{-i(p_{0{\rm i}} - p_0 - \hbar k_f - k[\sigma_{\rm ip} - (\ell'+j')\hbar])\cdot t_2/\hbar}\Big].
\end{aligned} \tag{25}$$

The integration of the second line in Eq. (25) gives $\mathbf{k} = 0$ if $\ell + j + \ell' + j' \neq 0$. To avoid this, $\ell + j + \ell' + j' = 0$ is required and the integration gives unity. Then, summing over ℓ' and using the time variable $T \equiv (t_1 + t_2)/2$ and $\Delta t \equiv t_2 - t_1$, we

obtain

$$\sum_\lambda \left\langle a^{\dagger\lambda}_{k_f} a^\lambda_{k_f} \right\rangle = \mathrm{Re}\Big[\frac{\mu_0 e^2}{8V\hbar}\int_{-\infty}^{\infty} dT \int_{-\infty}^{\infty} d\Delta t$$
$$\sum_{\ell,j,j'} \frac{1}{k_f p_i^0 p^0} M_{j,j'} C_\ell(\xi_{\rm ip}/\hbar, \eta_{\rm ip}/\hbar, \beta_{\rm ip}) C_{-\ell-j-j'}(\xi_{\rm pi}/\hbar, \eta_{\rm pi}/\hbar, \beta_{\rm ip})$$
$$\times\, e^{-i(p_{\rm i}-p-\hbar k_f - k[\sigma_{\rm ip}+(\ell+j)\hbar])\cdot\Delta t/\hbar}\Big]. \tag{26}$$

Integration with respect to Δt yields the Dirac's delta function $2\pi\hbar\delta(p_{0\rm i} - p_0 - \hbar k_f - k[\sigma_{\rm ip} + (\ell + j)\hbar])$.

The total radiation rate per unit time is given by

$$\frac{dE}{dT} = \frac{d}{dT}\sum_{k_f,\lambda} \hbar k_f \left\langle a^{\dagger\lambda}_{k_f} a^\lambda_{k_f} \right\rangle$$
$$= \mathrm{Re}\Big[\frac{\mu_0 \hbar e^2}{4(2\pi)^2}\int dk_f d^2\Omega_{k_f} \frac{k_f^2}{p_i^0} \sum_{\ell,j,j'} M_{j,j'} C_\ell(\xi_{\rm ip}/\hbar, \eta_{\rm ip}/\hbar, \beta_{\rm ip})$$
$$C_{-\ell-j-j'}(\xi_{\rm pi}/\hbar, \eta_{\rm pi}/\hbar, \beta_{\rm ip})\delta(p^2+m^2)\theta(p^0)\Big], \tag{27}$$

where $d^2\Omega_{k_f}$ is the solid angle in the Fourier space of the wavenumber of emitted photon k_f. In deriving Eq. (27), we used the relation

$$\frac{1}{2p^0} = \int_{-\infty}^{\infty} dp^0 \delta(p^2+m^2)\theta(p^0), \tag{28}$$

and performed the integration with respect to p^0. In expression (27), p_0 should be understood $p_0 = p_{0\rm i} - \hbar k_f - k[\sigma_{\rm ip} + (\ell + j)\hbar]$. Combined with the momentum conservation (24), we have the relation of the energy momentum conservation

$$p_\mu = p_{{\rm i}\mu} - \hbar k_{f\mu} - k_\mu[\sigma_{\rm ip} + \hbar(\ell + j)]. \tag{29}$$

In the present paper, we consider the following three cases of the initial state of the charged particle.

Case (A) First is the case when the initial momentum of the incident charged particle is specified by $p_{\rm i}^\mu = (m, 0, 0, 0)$. We choose the spatial coordinate $k^\mu = k(1, 0, 0, 1)$ and $k_f^\mu = k_f(1, \sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$ for all three cases. In this case, we have

$$p_{\rm i}\cdot k = -mk, \quad p_{\rm i}\cdot k_f = -mk_f, \quad k_f\cdot k = -2k_f k\sin^2\frac{\theta}{2}. \tag{30}$$

Case (B) Second is the case when the incident charged particle is in a relativistic motion, and we choose $p_{\rm i}^\mu = (np_i, p_i, 0, 0)$, where we defined $n = \sqrt{1 + m^2/p_i^2}$. In this case, we have

$$p_{\rm i}\cdot k = -np_{\rm i}k, \quad p_{\rm i}\cdot k_f = -p_{\rm i}k_f(n - \sin\theta\cos\phi), \quad k_f\cdot k = -2k_f k\sin^2\frac{\theta}{2}. \tag{31}$$

Case (C) Third is the case when the mean physical momentum of the charged particle is zero. Because the mean physical momentum is given by $q_{\rm i}^\mu = p_{\rm i}^\mu - (e^2 a\cdot a^*/4q_{\rm i}\cdot k)k^\mu$ [a], then we choose $q_{\rm i}^\mu = (m,0,0,0)$. In this case, we have

$$p_{\rm i}\cdot k = -mk, \quad p_{\rm i}\cdot k_f = -mk_f + \frac{e^2 a\cdot a^*}{2m}k_f \sin^2\frac{\theta}{2}, \quad k_f\cdot k = -2k_f k\sin^2\frac{\theta}{2}. \tag{32}$$

In the present paper, we consider the linear polarization for the electromagnetic wave background, and assume the constant vector $a^\mu = (0,a,0,0)$. Thus, we assume that the motion of the charged particle is the same direction of the polarization vector in case (B). Due to a fixed electromagnetic wave background, our system is not Lorentz invariant. Then, we treat three cases (A), (B), and (C) separately.

We consider the consequence of the relation (29), from which we have

$$\begin{aligned} -(p^2+m^2) &= 2\hbar p_{\rm i}\cdot k_f + 2\hbar(\ell+j)(p_{\rm i}\cdot k - \hbar k_f\cdot k) + 2\sigma_{\rm ip}(p_{\rm i}\cdot k - \hbar k_f\cdot k) \\ &= 2\hbar p_{\rm i}\cdot k_f + 2\hbar(\ell+j)(p_{\rm i}\cdot k - \hbar k_f\cdot k) - \frac{\hbar e^2 k_f\cdot k}{2p_{\rm i}\cdot k}a\cdot a^*. \end{aligned} \tag{33}$$

Because ${p_{\rm i}}_\mu$ is a time-like vector, then $p_{\rm i}\cdot k_f$ is negative. Similarly, because p_μ is time-like, then $p\cdot k$ is negative, which leads $p_i\cdot k - \hbar k_f\cdot k$ must be negative, using Eq. (29). Because k_μ and ${k_f}_\mu$ are null vectors, $k\cdot k_f$ is negative, then σ_{ip} is positive. Hence, from $p^2+m^2=0$ with Eq. (33), $\ell+j$ must be negative. Introducing the positive integer $r = -(\ell+j) \geq 1$, Eq. (29) is now written as $p_\mu = p_{{\rm i}\mu} - \hbar k_{f\mu} - k_\mu(\sigma_{\rm ip} - \hbar r)$.

From the relation $p^2+m^2=0$ with (33), we have

$$k_f = \begin{cases} \dfrac{rk}{1+\left(\nu^2 + 2r\frac{\hbar k}{m}\right)\sin^2\frac{\theta}{2}} & \text{for (A)}, \\ \dfrac{nrk}{(n-\sin\theta\cos\phi) + \left(\frac{m^2}{np_{\rm i}^2}\nu^2 + 2r\frac{\hbar k}{m}\frac{m}{p_{\rm i}}\right)\sin^2\frac{\theta}{2}} & \text{for (B)}, \\ \dfrac{rk}{1+2r\frac{\hbar k}{m}\sin^2\frac{\theta}{2}} & \text{for (C)}, \end{cases} \tag{34}$$

respectively, where we define the dimensionless parameter

$$\nu^2 = \frac{e^2}{2m^2}a\cdot a^*. \tag{35}$$

We may rewrite the expression in Eq.(27) as

$$\begin{aligned} &\sum_{\ell=-\infty}^{-j} C_\ell(\xi_{\rm ip}/\hbar, \eta_{\rm ip}/\hbar, \alpha_{\rm ip}, \beta_{\rm ip}) C_{-\ell-j-j'}(\xi_{\rm pi}/\hbar, \eta_{\rm pi}/\hbar, \alpha_{\rm ip}, \beta_{\rm ip}) \\ &\qquad = \sum_{r=1}^{\infty} C_{-r-j}(\xi_{\rm ip}/\hbar, \eta_{\rm ip}/\hbar, \alpha_{\rm ip}, \beta_{\rm ip}) C_{r-j'}(\xi_{\rm pi}/\hbar, \eta_{\rm pi}/\hbar, \alpha_{\rm ip}, \beta_{\rm ip}), \end{aligned} \tag{36}$$

[a] This expression of the mean momentum is read from the energy momentum conservation (29), as described in section 4.

then, by performing the integration with respect to k_f, we obtain

$$\frac{d^3E}{d\Omega_{k_f}dT} = \begin{cases} \frac{\mu_0 e^2}{8m^2(2\pi)^2}\sum_{j,j'=-2}^{2}\sum_{r=1}^{\infty}\frac{r^2k^2}{(1+(\nu^2+2r\frac{\hbar k}{m})\sin^2\frac{\theta}{2})^3} \\ \times M_{j,j'}C_{-r-j}(\xi_{\rm ip}/\hbar,\eta_{\rm ip}/\hbar,\alpha_{\rm ip},\beta_{\rm ip})C_{r-j'}(\xi_{\rm pi}/\hbar,\eta_{\rm pi}/\hbar,\alpha_{\rm ip},\beta_{\rm ip}) \\ \hfill \text{for (A)}, \\ \frac{\mu_0 e^2}{8m^2(2\pi)^2}\sum_{j,j'=-2}^{2}\sum_{r=1}^{\infty}\frac{n^2r^2k^2}{(n-\sin\theta\cos\phi+(\frac{m^2}{np_{\rm i}^2}\nu^2+2r\frac{\hbar k}{m}\frac{m}{p_{\rm i}})\sin^2\frac{\theta}{2})^3} \\ \times M_{j,j'}C_{-r-j}(\xi_{\rm ip}/\hbar,\eta_{\rm ip}/\hbar,\alpha_{\rm ip},\beta_{\rm ip})C_{r-j'}(\xi_{\rm pi}/\hbar,\eta_{\rm pi}/\hbar,\alpha_{\rm ip},\beta_{\rm ip}) \\ \hfill \text{for (B)}, \\ \frac{\mu_0 e^2}{8m^2(2\pi)^2}\sum_{j,j'=-2}^{2}\sum_{r=1}^{\infty}\frac{r^2k^2}{(1+2r\frac{\hbar k}{m}\sin^2\frac{\theta}{2})^3} \\ \times M_{j,j'}C_{-r-j}(\xi_{\rm ip}/\hbar,\eta_{\rm ip}/\hbar,\alpha_{\rm ip},\beta_{\rm ip})C_{r-j'}(\xi_{\rm pi}/\hbar,\eta_{\rm pi}/\hbar,\alpha_{\rm ip},\beta_{\rm ip}) \\ \hfill \text{for (C)}, \end{cases} \tag{37}$$

respectively, where k_f should be regarded as the expression of the right-hand-side of (34). Here we omit the step function $\theta(p^0)$ because of the time-like property of p^μ. When the background electromagnetic wave is linearly polarized and the constant vector is $a^\mu = (0, a, 0, 0)$, we have

$$\frac{\sigma_{\rm ip}}{\hbar} = \frac{\eta_{\rm ip}}{\hbar} = -\frac{\eta_{\rm pi}}{\hbar} = \begin{cases} \frac{r\nu^2\sin^2\frac{\theta}{2}}{1+\nu^2\sin^2\frac{\theta}{2}} & \text{for (A)}, \\ \frac{r\nu^2\sin^2\frac{\theta}{2}}{n\frac{p_{\rm i}^2}{m^2}(n-\sin\theta\cos\phi)+\nu^2\sin^2\frac{\theta}{2}} & \text{for (B)}, \\ r\nu^2\sin^2\frac{\theta}{2} & \text{for (C)}, \end{cases} \tag{38}$$

and

$$\frac{\xi_{\rm ip}}{\hbar} = -\frac{\xi_{\rm pi}}{\hbar} = \begin{cases} -\frac{\sqrt{2}r\nu\sin\theta\cos\phi}{1+\nu^2\sin^2\frac{\theta}{2}} & \text{for (A)}, \\ -\frac{\sqrt{2}r\frac{m}{p_{\rm i}}\nu n\left(2\sin^2\frac{\theta}{2}-\sin\theta\cos\phi\right)}{n(n-\sin\theta\cos\phi)+\frac{m^2}{p_{\rm i}^2}\nu^2\sin^2\frac{\theta}{2}} & \text{for (B)}, \\ -\sqrt{2}r\nu\sin\theta\cos\phi & \text{for (C)}, \end{cases} \tag{39}$$

respectively. It is worthy to note that the expression (36) does not contain $\hbar$.

3. results

In this section, we present the explicit formulas for the radiation in three cases, then discuss about the origin of the quantum effect by extracting the term of the order of $\hbar$.

3.1. $p_i^\mu = (m, 0, 0, 0)$

First let us consider *case* (A). Explicit expression of the kernel $M_{i,j}$ is summarized in Appendix A. Expression (37) yields

$$
\begin{aligned}
\frac{d^3E}{d\Omega_{k_f}dT} &= \frac{\mu_0 e^2}{8(2\pi)^2}\sum_{s,s'=-\infty}^{\infty}\sum_{r=1}^{\infty}\frac{r^2k^2}{(1+(\nu^2+2r\frac{\hbar k}{m})\sin^2\frac{\theta}{2})^3}\\
&\times\Big\{-4\frac{1+\nu^2+\left(\nu^4+\frac{\hbar kr}{m}(2+\frac{\hbar kr}{m})+\nu^2(1+2\frac{\hbar kr}{m})\right)\sin^2\frac{\theta}{2}}{1+(\nu^2+2\frac{\hbar kr}{m})\sin^2\frac{\theta}{2}}\\
&\times J_{-r-2s}(\xi_{\rm ip}/\hbar)J_s(\eta_{\rm ip}/2\hbar)J_{r-2s'}(\xi_{\rm pi}/\hbar)J_{s'}(\eta_{\rm pi}/2\hbar)\\
&+\frac{\sqrt{2}\nu\frac{\hbar^2k^2r^2}{m^2}\cos\phi\sin^2\frac{\theta}{2}\sin\theta}{\left(1+\nu^2\sin^2\frac{\theta}{2}\right)\left(1+\left(\nu^2+2\frac{\hbar kr}{m}\right)\sin^2\frac{\theta}{2}\right)}\\
&\times\Big(J_{-r-1-2s}(\xi_{\rm ip}/\hbar)J_s(\eta_{\rm ip}/2\hbar)J_{r-2s'}(\xi_{\rm pi}/\hbar)J_{s'}(\eta_{\rm pi}/2\hbar)\\
&+J_{-r+1-2s}(\xi_{\rm ip}/\hbar)J_s(\eta_{\rm ip}/2\hbar)J_{r-2s'}(\xi_{\rm pi}/\hbar)J_{s'}(\eta_{\rm pi}/2\hbar)\\
&+J_{-r-2s}(\xi_{\rm ip}/\hbar)J_s(\eta_{\rm ip}/2\hbar)J_{r-1-2s'}(\xi_{\rm pi}/\hbar)J_{s'}(\eta_{\rm pi}/2\hbar)\\
&+J_{-r-2s}(\xi_{\rm ip}/\hbar)J_s(\eta_{\rm ip}/2\hbar)J_{r+1-2s'}(\xi_{\rm pi}/\hbar)J_{s'}(\eta_{\rm pi}/2\hbar)\Big)\\
&+2\nu^2\Big(J_{-r-1-2s}(\xi_{\rm ip}/\hbar)J_s(\eta_{\rm ip}/2\hbar)J_{r-1-2s'}(\xi_{\rm pi}/\hbar)J_{s'}(\eta_{\rm pi}/2\hbar)\\
&+J_{-r+1-2s}(\xi_{\rm ip}/\hbar)J_s(\eta_{\rm ip}/2\hbar)J_{r+1-2s'}(\xi_{\rm pi}/\hbar)J_{s'}(\eta_{\rm pi}/2\hbar)\\
&+J_{-r-1-2s}(\xi_{\rm ip}/\hbar)J_s(\eta_{\rm ip}/2\hbar)J_{r+1-2s'}(\xi_{\rm pi}/\hbar)J_{s'}(\eta_{\rm pi}/2\hbar)\\
&+J_{-r+1-2s}(\xi_{\rm ip}/\hbar)J_s(\eta_{\rm ip}/2\hbar)J_{r-1-2s'}(\xi_{\rm pi}/\hbar)J_{s'}(\eta_{\rm pi}/2\hbar)\Big)\\
&-\frac{\nu^2\left(1+\left(\nu^2+\frac{\hbar kr}{m}\right)\sin^2\frac{\theta}{2}\right)^2}{\left(1+\nu^2\sin^2\frac{\theta}{2}\right)\left(1+\left(\nu^2+2\frac{\hbar kr}{m}\right)\sin^2\frac{\theta}{2}\right)}\\
&\times\Big(J_{-r-2-2s}(\xi_{\rm ip}/\hbar)J_s(\eta_{\rm ip}/2\hbar)J_{r-2s'}(\xi_{\rm pi}/\hbar)J_{s'}(\eta_{\rm pi}/2\hbar)\\
&+J_{-r+2-2s}(\xi_{\rm ip}/\hbar)J_s(\eta_{\rm ip}/2\hbar)J_{r-2s'}(\xi_{\rm pi}/\hbar)J_{s'}(\eta_{\rm pi}/2\hbar)\\
&+J_{-r-2s}(\xi_{\rm ip}/\hbar)J_s(\eta_{\rm ip}/2\hbar)J_{r-2-2s'}(\xi_{\rm pi}/\hbar)J_{s'}(\eta_{\rm pi}/2\hbar)\\
&+J_{-r-2s}(\xi_{\rm ip}/\hbar)J_s(\eta_{\rm ip}/2\hbar)J_{r+2-2s'}(\xi_{\rm pi}/\hbar)J_{s'}(\eta_{\rm pi}/2\hbar)\Big)\Big\}.
\end{aligned}
\tag{40}
$$

When the background field is weak, i.e., $\nu \ll 1$, one can verify that $r = 1$ gives the dominant contribution. In this case, we find the first order quantum effect by expanding the above expression up to $O(\nu^2)$ and $O(\hbar)$,

$$\frac{d^3E}{d\Omega_{k_f}dT} = \frac{\mu_0 e^2k^2\nu^2}{(4\pi)^2}(1-6\frac{\hbar k}{m}\sin^2\frac{\theta}{2})\left(1-\cos^2\phi\sin^2\theta\right). \tag{41}$$

The integration with respect to Ω_{k_f} gives

$$\frac{dE}{dT} = \frac{\mu_0 e^2k^2\nu^2}{6\pi}(1-3\frac{\hbar k}{m}), \tag{42}$$

which means that the first-order quantum effect decreases the radiation energy.

3.2. $p_i^\mu = (np_i, p_i, 0, 0)$

In *case* (B), expression (37) leads to

$$\begin{aligned}
\frac{d^3E}{d\Omega_{k_f}dT} &= \frac{\mu_0 e^2}{8(2\pi)^2}\sum_{s,s'=-\infty}^{\infty}\sum_{r=1}^{\infty}\frac{n^2r^2k^2}{\left((n-\sin\theta\cos\phi)+(\frac{m^2}{np_{\rm i}^2}\nu^2+2r\frac{\hbar k}{m}\frac{m}{p_{\rm i}})\sin^2\frac{\theta}{2}\right)^3}\\
&\times\left\{-4\frac{n(1+\nu^2)(n-\sin\theta\cos\phi)+\left(\frac{m^2}{p_{\rm i}^2}(\nu^2+\nu^4)+2n\frac{\hbar kr}{m}\frac{m}{p_{\rm i}}(1+\nu^2)+n^2\frac{\hbar^2k^2r^2}{m^2}\right)\sin^2\frac{\theta}{2}}{n(n-\sin\theta\cos\phi)+\left(\frac{m^2}{p_{\rm i}^2}\nu^2+2n\frac{\hbar kr}{m}\frac{m}{p_{\rm i}}\right)\sin^2\frac{\theta}{2}}\right.\\
&\times J_{-r-2s}(\xi_{{\rm i}p}/\hbar)J_s(\eta_{{\rm i}p}/2\hbar)J_{r-2s'}(\xi_{p{\rm i}}/\hbar)J_{s'}(\eta_{p{\rm i}}/2\hbar)\\
&+\sqrt{2}\frac{n^3\nu\frac{\hbar^2k^2r^2}{m^2}\frac{m}{p_{\rm i}}\sin^2\frac{\theta}{2}\left(2\sin^2\frac{\theta}{2}-\sin\theta\cos\phi\right)}{\left(n(n-\sin\theta\cos\phi)+\left(\frac{m^2}{p_{\rm i}^2}\nu^2+2n\frac{\hbar kr}{m}\frac{m}{p_{\rm i}}\right)\sin^2\frac{\theta}{2}\right)\left(n(n-\sin\theta\cos\phi)+\frac{m^2}{p_{\rm i}^2}\nu^2\sin^2\frac{\theta}{2}\right)}\\
&\times\Big(J_{-r-1-2s}(\xi_{{\rm i}p}/\hbar)J_s(\eta_{{\rm i}p}/2\hbar)J_{r-2s'}(\xi_{p{\rm i}}/\hbar)J_{s'}(\eta_{p{\rm i}}/2\hbar)\\
&+J_{-r+1-2s}(\xi_{{\rm i}p}/\hbar)J_s(\eta_{{\rm i}p}/2\hbar)J_{r-2s'}(\xi_{p{\rm i}}/\hbar)J_{s'}(\eta_{p{\rm i}}/2\hbar)\\
&+J_{-r-2s}(\xi_{{\rm i}p}/\hbar)J_s(\eta_{{\rm i}p}/2\hbar)J_{r-1-2s'}(\xi_{p{\rm i}}/\hbar)J_{s'}(\eta_{p{\rm i}}/2\hbar)\\
&+J_{-r-2s}(\xi_{{\rm i}p}/\hbar)J_s(\eta_{{\rm i}p}/2\hbar)J_{r+1-2s'}(\xi_{p{\rm i}}/\hbar)J_{s'}(\eta_{p{\rm i}}/2\hbar)\Big)\\
&+2\nu^2\Big(J_{-r-1-2s}(\xi_{{\rm i}p}/\hbar)J_s(\eta_{{\rm i}p}/2\hbar)J_{r-1-2s'}(\xi_{p{\rm i}}/\hbar)J_{s'}(\eta_{p{\rm i}}/2\hbar)\\
&+J_{-r+1-2s}(\xi_{{\rm i}p}/\hbar)J_s(\eta_{{\rm i}p}/2\hbar)J_{r+1-2s'}(\xi_{p{\rm i}}/\hbar)J_{s'}(\eta_{p{\rm i}}/2\hbar)\\
&+J_{-r-1-2s}(\xi_{{\rm i}p}/\hbar)J_s(\eta_{{\rm i}p}/2\hbar)J_{r+1-2s'}(\xi_{p{\rm i}}/\hbar)J_{s'}(\eta_{p{\rm i}}/2\hbar)\\
&+J_{-r+1-2s}(\xi_{{\rm i}p}/\hbar)J_s(\eta_{{\rm i}p}/2\hbar)J_{r-1-2s'}(\xi_{p{\rm i}}/\hbar)J_{s'}(\eta_{p{\rm i}}/2\hbar)\Big)\\
&-\frac{\nu^2}{4}\frac{\left(2n(n-\sin\theta\cos\phi)+\frac{m^2}{p_{\rm i}^2}\nu^2+n\frac{\hbar kr}{m}\frac{m}{p_{\rm i}}-\left(\frac{m^2}{p_{\rm i}^2}\nu^2+n\frac{\hbar kr}{m}\frac{m}{p_{\rm i}}\right)\cos\theta\right)^2}{\left(n(n-\sin\theta\cos\phi)+\frac{m^2}{p_{\rm i}^2}\nu^2\sin^2\frac{\theta}{2}\right)\left(n(n-\sin\theta\cos\phi)+\left(\frac{m^2}{p_{\rm i}^2}\nu^2+2n\frac{\hbar kr}{m}\frac{m}{p_{\rm i}}\right)\sin^2\frac{\theta}{2}\right)}\\
&\times\Big(J_{-r-2-2s}(\xi_{{\rm i}p}/\hbar)J_s(\eta_{{\rm i}p}/2\hbar)J_{r-2s'}(\xi_{p{\rm i}}/\hbar)J_{s'}(\eta_{p{\rm i}}/2\hbar)\\
&+J_{-r+2-2s}(\xi_{{\rm i}p}/\hbar)J_s(\eta_{{\rm i}p}/2\hbar)J_{r-2s'}(\xi_{p{\rm i}}/\hbar)J_{s'}(\eta_{p{\rm i}}/2\hbar)\\
&+J_{-r-2s}(\xi_{{\rm i}p}/\hbar)J_s(\eta_{{\rm i}p}/2\hbar)J_{r-2-2s'}(\xi_{p{\rm i}}/\hbar)J_{s'}(\eta_{p{\rm i}}/2\hbar)\\
&\left.+J_{-r-2s}(\xi_{{\rm i}p}/\hbar)J_s(\eta_{{\rm i}p}/2\hbar)J_{r+2-2s'}(\xi_{p{\rm i}}/\hbar)J_{s'}(\eta_{p{\rm i}}/2\hbar)\Big)\right\}. \qquad (43)
\end{aligned}$$

Expanding this expression up to $O(\nu^2)$, $O(\hbar)$ and $O(m/p_{\rm i})$ in the relativistic limit, we obtain

$$\frac{d^3E}{d\Omega_{k_f}dT} = \frac{\mu_0 k^2\nu^2e^2}{4(2\pi)^2}\left(\frac{1}{(1-\sin\theta\cos\phi)^3}-6\frac{\hbar k}{m}\frac{m}{p_{\rm i}}\frac{\sin^2\frac{\theta}{2}}{(1-\sin\theta\cos\phi)^4}\right). \qquad (44)$$

The integration with respect to Ω_{k_f} gives

$$\frac{dE}{dT} = \frac{\mu_0k^2e^2\nu^2}{32\pi}\left(3-7\frac{\hbar k}{m}\frac{m}{p_{\rm i}}\right). \qquad (45)$$

3.3. $q_i^\mu = (m, 0, 0, 0)$

In *case* (C), expression (37) yields

$$\frac{d^3E}{d\Omega_{k_f}dT} = \frac{\mu_0 e^2}{8(2\pi)^2}\sum_{s,s'=-\infty}^{\infty}\sum_{r=1}^{\infty}\frac{r^2k^2}{(1+2r\frac{\hbar k}{m}\sin^2\frac{\theta}{2})^3}$$
$$\times\Big\{-4\frac{1+\nu^2+\frac{\hbar kr}{m}(2(1+\nu^2)+\frac{\hbar kr}{m})\sin^2\frac{\theta}{2}}{1+2\frac{\hbar kr}{m}\sin^2\frac{\theta}{2}}$$
$$\times J_{-r-2s}(\xi_{\rm ip}/\hbar)J_s(\eta_{\rm ip}/2\hbar)J_{r-2s'}(\xi_{\rm pi}/\hbar)J_{s'}(\eta_{\rm pi}/2\hbar)+\frac{\sqrt{2}\nu\frac{\hbar^2k^2r^2}{m^2}\cos\phi\sin^2\frac{\theta}{2}\sin\theta}{1+2\frac{\hbar kr}{m}\sin^2\frac{\theta}{2}}$$
$$\times\Big(J_{-r-1-2s}(\xi_{\rm ip}/\hbar)J_s(\eta_{\rm ip}/2\hbar)J_{r-2s'}(\xi_{\rm pi}/\hbar)J_{s'}(\eta_{\rm pi}/2\hbar)$$
$$+J_{-r+1-2s}(\xi_{\rm ip}/\hbar)J_s(\eta_{\rm ip}/2\hbar)J_{r-2s'}(\xi_{\rm pi}/\hbar)J_{s'}(\eta_{\rm pi}/2\hbar)$$
$$+J_{-r-2s}(\xi_{\rm ip}/\hbar)J_s(\eta_{\rm ip}/2\hbar)J_{r-1-2s'}(\xi_{\rm pi}/\hbar)J_{s'}(\eta_{\rm pi}/2\hbar)$$
$$+J_{-r-2s}(\xi_{\rm ip}/\hbar)J_s(\eta_{\rm ip}/2\hbar)J_{r+1-2s'}(\xi_{\rm pi}/\hbar)J_{s'}(\eta_{\rm pi}/2\hbar)\Big)$$
$$+2\nu^2\Big(J_{-r-1-2s}(\xi_{\rm ip}/\hbar)J_s(\eta_{\rm ip}/2\hbar)J_{r-1-2s'}(\xi_{\rm pi}/\hbar)J_{s'}(\eta_{\rm pi}/2\hbar)$$
$$+J_{-r+1-2s}(\xi_{\rm ip}/\hbar)J_s(\eta_{\rm ip}/2\hbar)J_{r+1-2s'}(\xi_{\rm pi}/\hbar)J_{s'}(\eta_{\rm pi}/2\hbar)$$
$$+J_{-r-1-2s}(\xi_{\rm ip}/\hbar)J_s(\eta_{\rm ip}/2\hbar)J_{r+1-2s'}(\xi_{\rm pi}/\hbar)J_{s'}(\eta_{\rm pi}/2\hbar)$$
$$+J_{-r+1-2s}(\xi_{\rm ip}/\hbar)J_s(\eta_{\rm ip}/2\hbar)J_{r-1-2s'}(\xi_{\rm pi}/\hbar)J_{s'}(\eta_{\rm pi}/2\hbar)\Big)$$
$$-\frac{\nu^2\left(1+\frac{\hbar kr}{m}\sin^2\frac{\theta}{2}\right)^2}{1+2\frac{\hbar kr}{m}\sin^2\frac{\theta}{2}}\Big(J_{-r-2-2s}(\xi_{\rm ip}/\hbar)J_s(\eta_{\rm ip}/2\hbar)J_{r-2s'}(\xi_{\rm pi}/\hbar)J_{s'}(\eta_{\rm pi}/2\hbar)$$
$$+J_{-r+2-2s}(\xi_{\rm ip}/\hbar)J_s(\eta_{\rm ip}/2\hbar)J_{r-2s'}(\xi_{\rm pi}/\hbar)J_{s'}(\eta_{\rm pi}/2\hbar)$$
$$+J_{-r-2s}(\xi_{\rm ip}/\hbar)J_s(\eta_{\rm ip}/2\hbar)J_{r-2-2s'}(\xi_{\rm pi}/\hbar)J_{s'}(\eta_{\rm pi}/2\hbar)$$
$$+J_{-r-2s}(\xi_{\rm ip}/\hbar)J_s(\eta_{\rm ip}/2\hbar)J_{r+2-2s'}(\xi_{\rm pi}/\hbar)J_{s'}(\eta_{\rm pi}/2\hbar)\Big)\Big\}. \tag{46}$$

Expanding up to $O(\hbar)$ and $O(\nu^2)$, we find

$$\frac{d^3E}{d\Omega_{k_f}dT} = \frac{\mu_0 k^2\nu^2e^2}{16(2\pi)^2}(3+2\cos 2\theta\cos^2\phi-\cos 2\phi)(1+6\frac{\hbar k}{m}\sin^2\frac{\theta}{2}). \tag{47}$$

The integration with respect to Ω_{k_f} gives

$$\frac{dE}{dT} = \frac{\mu_0 k^2e^2\nu^2}{12\pi}\left(1-3\frac{\hbar k}{m}\right). \tag{48}$$

Let us summarize the results in this section. We presented the explicit formulas for the radiation rate for three cases. In all three cases, the first order quantum effect decreases the total radiation energy. Because $\xi_{ip}/\hbar$ and $\eta_{ip}/\hbar$ do not depend on $\hbar$, then the Bessel functions in the radiation formulas do not contain $\hbar$. As demonstrated in appendix A, the quantum correction in the kernel $M_{i,j}$ arises at the order of $\hbar^2$. Then, the first order correction of $\hbar$ comes from the first line of the radiation formula (40), (43) and (46), respectively.

4. summary and conclusions

In the present paper, we investigated the quantum effect on the Larmor radiation from a moving charge in a monochromatic electromagnetic plane wave based on the SQED. Our work is different from the previous works in the following points. First, we derived the theoretical formula starting with the framework of the in-in formalism[13]. Second, we demonstrated that the quantum effect generally suppresses the total radiation energy by explicitly evaluating the contribution at the order of $\hbar$. To this end, we considered the three cases for the initial state, (A) the initial momentum of the charged particle is zero, (B) the charged particle is in the relativistic motion along with the direction of the linear polarization of electromagnetic plane wave, and (C) the mean momentum is zero. We derived the formula for the radiation per unit time (40), (43) and (46), for (A), (B) and (C), respectively. The results contain the two parameters, the strength of the electromagnetic plane wave ν and the frequency of the electromagnetic plane wave $\hbar k/m$ for (A) and (C). One more parameter, the initial momentum of the charged particle m/p_i is added for (B). We find that the first order quantum correction is the order of $\hbar k/m$ for (A) and (C), and is the order of $\hbar k/m$ multiplied by the additional factor m/p_i for (B). Thus, the first order quantum effect is the order $\hbar k/m$ at most.

The quantum effect decreases the total radiation energy compared with the classical radiation formula for three cases. Let us discuss about the origin of the quantum effect. As described in the previous sections, the quantum effect in M_{ij} appears at the order of $\hbar^2$ and the Bessel function doesn't contain $\hbar$ in their arguments. The first order quantum correction comes from the denominator of the factor in the first line of Eqs. (40), (43) and (46), for (A), (B) and (C), respectively. This factor comes from the scattered photon energy (34) and the energy-momentum conservation (29), which is rewritten as,

$$p_\mu - \frac{e^2|a|^2}{4p\cdot k}k_\mu + \hbar {k_f}_\mu = p_{\mathrm{i}\mu} - \frac{e^2|a|^2}{4p_i\cdot k}k_\mu + \hbar r k_\mu. \tag{49}$$

This represents the energy-momentum conservation of the (nonlinear) Compton scattering as $p_\mu - (e^2|a|^2/4p\cdot k)k_\mu$ is regarded as the mean momentum of charged particle averaged over long duration of time. Therefore, the first order quantum effect originates *only* from the inelastic collision of the Compton scattering. This explains the decrease of the energy of a scattered photon and the total radiation energy. Thus, the particle properties of a photon is the origin of the quantum effect in the case of the present paper.

Let us discuss the differences between the results in the present paper and those in the previous works[4,1,2,3]. First, in the previous works, the first order quantum effect is identified as the non-local nature of the charged particle, which is a sharp contrast to the conclusion of the present work. The previous works considered a spatially homogeneous electric field background, in which we could not regard the radiation process as a collision between a charged particle and background photons. Second, in our previous work[2] based on the Wentzel-Kramers-Brillouin (WKB) ap-

proximation, the quantum effect may increase the total radiation when the charged particle is in the relativistic motion in an oscillating homogeneous electric field. This difference of the results can be ascribed to the difference of the origin of the quantum effect.

Our theoretical framework, based on the SQED, might be useful as a tool to investigate the quantum radiation from an electron in an intense electromagnetic field. When we assume a charged particle as an electron and the electromagnetic plane wave background as an X-ray laser, we have

$$\frac{\hbar k}{m} \simeq 2 \times 10^{-3} \left(\frac{\hbar k}{1\mathrm{eV}}\right)\left(\frac{0.5\mathrm{MeV}}{m}\right). \tag{50}$$

Thus, the quantum effect is very small. This will make it hard to detect the quantum effect in an experiment. However, the interaction between the laser and an electron has attracted many researchers even in the area of the fundamental physics (see e.g., Refs. 22, 23 and references therein). Further investigations on the topic, how the quantum effect could be tested in experiments, are left as a future problem.

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Appendix A. Expression of kernel

In the appendix, we summarize the explicit expression of the kernel M_{ij}. Straightforward calculation leads to

$$M_{0,0} = -4m^2 \frac{1+\nu^2+\left(\nu^4+\frac{\hbar kr}{m}(2+\frac{\hbar kr}{m})+\nu^2(1+2\frac{\hbar kr}{m})\right)\sin^2\frac{\theta}{2}}{1+(\nu^2+2\frac{\hbar kr}{m})\sin^2\frac{\theta}{2}}$$

$$M_{1,0} = M_{0,1} = M_{-1,0} = M_{0,-1} = \sqrt{2}m^2 \frac{\nu\frac{\hbar^2k^2r^2}{m^2}\cos\phi\sin^2\frac{\theta}{2}\sin\theta}{\left(1+\nu^2\sin^2\frac{\theta}{2}\right)\left(1+\left(\nu^2+2\frac{\hbar kr}{m}\right)\sin^2\frac{\theta}{2}\right)}$$

$$M_{1,1} = M_{-1,-1} = M_{-1,1} = M_{1,-1} = 2m^2\nu^2$$

$$M_{2,0} = M_{0,2} = M_{-2,0} = M_{0,-2} = -m^2\nu^2 \frac{\left(1+\left(\nu^2+\frac{\hbar kr}{m}\right)\sin^2\frac{\theta}{2}\right)^2}{\left(1+\nu^2\sin^2\frac{\theta}{2}\right)\left(1+\left(\nu^2+2\frac{\hbar kr}{m}\right)\sin^2\frac{\theta}{2}\right)}$$

for (A),

$$
\begin{aligned}
M_{0,0} &= -4m^2 \frac{n(1+\nu^2)(n-\sin\theta\cos\phi) + \left(\frac{m^2}{p_{\rm i}^2}(\nu^2+\nu^4) + 2n\frac{\hbar kr}{m}\frac{m}{p_{\rm i}}(1+\nu^2) + n^2\frac{\hbar^2k^2r^2}{m^2}\right)\sin^2\frac{\theta}{2}}{n(n-\sin\theta\cos\phi) + \left(\frac{m^2}{p_{\rm i}^2}\nu^2 + 2n\frac{\hbar kr}{m}\frac{m}{p_{\rm i}}\right)\sin^2\frac{\theta}{2}} \\
M_{1,0} &= M_{0,1} = M_{-1,0} = M_{0,-1} \\
&= \sqrt{2}m^2 \frac{n^3\nu\frac{\hbar^2k^2r^2}{m^2}\frac{m}{p_{\rm i}}\sin^2\frac{\theta}{2}\left(2\sin^2\frac{\theta}{2} - \sin\theta\cos\phi\right)}{\left(n(n-\sin\theta\cos\phi) + \left(\frac{m^2}{p_{\rm i}^2}\nu^2 + 2n\frac{\hbar kr}{m}\frac{m}{p_{\rm i}}\right)\sin^2\frac{\theta}{2}\right)\left(n(n-\sin\theta\cos\phi) + \frac{m^2}{p_{\rm i}^2}\nu^2\sin^2\frac{\theta}{2}\right)} \\
M_{1,1} &= M_{-1,-1} = M_{-1,1} = M_{1,-1} = 2m^2\nu^2 \\
M_{2,0} &= M_{0,2} = M_{-2,0} = M_{0,-2} \\
&= -m^2\nu^2 \frac{\left(n(n-\sin\theta\cos\phi) + \left(\frac{m^2}{p_{\rm i}^2}\nu^2 + n\frac{\hbar kr}{m}\frac{m}{p_{\rm i}}\right)\sin^2\frac{\theta}{2}\right)^2}{\left(n(n-\sin\theta\cos\phi) + \frac{m^2}{p_{\rm i}^2}\nu^2\sin^2\frac{\theta}{2}\right)\left(n(n-\sin\theta\cos\phi) + \left(\frac{m^2}{p_{\rm i}^2}\nu^2 + 2n\frac{\hbar kr}{m}\frac{m}{p_{\rm i}}\right)\sin^2\frac{\theta}{2}\right)}
\end{aligned}
$$

for (B), and

$$
\begin{aligned}
M_{0,0} &= -4m^2 \frac{1+\nu^2 + \frac{\hbar kr}{m}\left(2(1+\nu^2) + \frac{\hbar kr}{m}\right)\sin^2\frac{\theta}{2}}{1 + 2\frac{\hbar kr}{m}\sin^2\frac{\theta}{2}} \\
M_{1,0} &= M_{0,1} = M_{-1,0} = M_{0,-1} = \sqrt{2}m^2\nu \frac{\frac{\hbar^2k^2r^2}{m^2}\sin\theta\cos\phi\sin^2\frac{\theta}{2}}{1 + 2\frac{\hbar kr}{m}\sin^2\frac{\theta}{2}} \\
M_{1,1} &= M_{-1,-1} = M_{-1,1} = M_{1,-1} = 2m^2\nu^2 \\
M_{2,0} &= M_{0,2} = M_{-2,0} = M_{0,-2} = -m^2 \frac{\nu^2\left(1 + \frac{\hbar kr}{m}\sin^2\frac{\theta}{2}\right)^2}{1 + 2\frac{\hbar kr}{m}\sin^2\frac{\theta}{2}}
\end{aligned}
$$

for (C), respectively. It is worthy to note the expressions of $M_{i,j}$ expanded in terms of $\hbar$. From the above formulas, we have

$$
\begin{aligned}
M_{0,0} &= -4m^2(1+\nu^2) - 4m^2\frac{\hbar^2k^2r^2}{m^2}\frac{\sin^2\frac{\theta}{2}}{1+\nu^2\sin^2\frac{\theta}{2}} + O(\hbar^3) \\
M_{1,0} &= M_{0,1} = M_{-1,0} = M_{0,-1} = \sqrt{2}m^2\nu\frac{\hbar^2k^2r^2}{m^2}\frac{\sin\theta\cos\phi\sin^2\frac{\theta}{2}}{\left(1+\nu^2\sin^2\frac{\theta}{2}\right)^2} + O(\hbar^3) \\
M_{1,1} &= M_{-1,-1} = M_{-1,1} = M_{1,-1} = 2m^2\nu^2 \\
M_{2,0} &= M_{0,2} = M_{-2,0} = M_{0,-2} = -m^2\nu^2 - m^2\frac{\hbar^2k^2r^2}{m^2}\frac{\nu^2\sin^4\frac{\theta}{2}}{\left(1+\nu^2\sin^2\frac{\theta}{2}\right)} + O(\hbar^3)
\end{aligned}
$$

for (A),

$$M_{0,0} = -4m^2(1+\nu^2) - 4m^2\frac{\hbar^2k^2r^2}{m^2}\frac{n^2\sin^2\frac{\theta}{2}}{n(n-\sin\theta\cos\phi)+\frac{m^2}{p_{\rm i}^2}\nu^2\sin^2\frac{\theta}{2}} + O(\hbar^3)$$

$$M_{1,0} = M_{0,1} = M_{-1,0} = M_{0,-1}$$

$$= \sqrt{2}m^2\frac{\hbar^2k^2r^2}{m^2}\frac{n^3\nu\frac{m}{p_{\rm i}}\sin^2\frac{\theta}{2}\left(2\sin^2\frac{\theta}{2}-\sin\theta\cos\phi\right)}{\left(n(n-\sin\theta\cos\phi)+\frac{m^2}{p_{\rm i}^2}\nu^2\sin^2\frac{\theta}{2}\right)^2} + O(\hbar^3)$$

$$M_{1,1} = M_{-1,-1} = M_{-1,1} = M_{1,-1} = 2m^2\nu^2$$

$$M_{2,0} = M_{0,2} = M_{-2,0} = M_{0,-2} =$$

$$= -m^2\nu^2 - m^2\frac{\hbar^2k^2r^2}{m^2}\frac{n^2\nu^2\sin^4\frac{\theta}{2}}{\left(n(n-\sin\theta\cos\phi)+\frac{m^2}{p_{\rm i}^2}\nu^2\sin^2\frac{\theta}{2}\right)^2} + O(\hbar^3)$$

for (B), and

$$M_{0,0} = -4m^2(1+\nu^2) - 4m^2\frac{\hbar^2k^2r^2}{m^2}\sin^2\frac{\theta}{2} + O(\hbar^3)$$

$$M_{1,0} = M_{0,1} = M_{-1,0} = M_{0,-1} = \sqrt{2}m^2\nu\frac{\hbar^2k^2r^2}{m^2}\sin\theta\cos\phi\sin^2\frac{\theta}{2} + O(\hbar^3)$$

$$M_{1,1} = M_{-1,-1} = M_{-1,1} = M_{1,-1} = 2m^2\nu^2$$

$$M_{2,0} = M_{0,2} = M_{-2,0} = M_{0,-2} = -m^2\nu^2 - m^2\frac{\hbar^2k^2r^2}{m^2}\nu^2\sin^4\frac{\theta}{2} + O(\hbar^3),$$

for (C), respectively. Thus, the quantum correction in the kernel $M_{i,j}$ arises at the order of $\hbar^2$.

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